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On variational approaches in NRT continua

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Abstract

In the paper some features of the theory of Not Resisting Tension (NRT) material are deepened. In details, one first introduces the basic NRT model, which is proved to simply and effectively interpreting the behaviour of mechanical bodies made by not-cohesive materials; thereafter one analyses energetic approaches and limit analysis tools for problems relevant to NRT continua. Afterward, on the basis of the fundamental variational theorems, the main rules governing the NRT behaviour are demonstrated, by imposing Kuhn–Tucker stationarity conditions for the stated constrained optimisation procedures. Finally an application is operated of the presented theory to an elastic NRT semi-plane subject to a distributed load, reproducing the stress situation induced in the soil by a foundation structure.

Keywords: Bi-dimensional statics; NRT model; Foundation structures; Energetic approach; Stress field; Stress propagation

1. The NRT material

1.1. General features of NRT behaviour

The *Not Resisting Tension* (NRT) material (Baratta, 1991; Di Pasquale, 1984; Bazant, 1996a) is a simple and complete phenomenological model for interpreting the mechanical bodies made by not-cohesive materials.

The theory of the Not Resisting Tension material, originally formulated by Heyman (Heyman, 1966; Heyman, 1969), is based on two fundamental hypotheses: the assumption of zero tensile resistance and the hypothesis of linear elastic behaviour in compression.

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Even as an idealization, the NRT model gives a pretty reliable representation of the real behaviour of materials exhibiting light tensile resistance, such as masonries or soils (Baratta, 1984, 1996; Cambou and Di Prisco, 2002; Broberg, 2000; Li and Bazant, 1996; Bazant, 1995; Acker et al., 1998).

The NRT material is usually conceived by assuming an elastic response, possibly non linear, along planes characterized by pure compression stresses, while one admits the development of free deformations (*fracture strains*) without energy dissipation along the other directions. That is to say that the NRT material is essentially a non-linear elastic material, whose non-linearity is mainly due to the development of fractures, which are absent in the compressive phase.

The NRT constitutive law is, then, usually schematised by an elastic stress–strain relation, which is unilateral, that is to say valid with reference to purely compressive fields.

Therefore, the domain of admissible stresses coincides with the Rankine's square, with infinite limit tension in compression.

By assuming negative values for compressive stresses, at the generic point P (Fig. 1) the stress tensor $\boldsymbol{\sigma}$ should be characterized by stress values along the principal directions '1' and '2' satisfying admissibility conditions; that is to say that, imposing $\sigma_2 \leqslant \sigma_1$, admissible principal stress states should satisfy the condition $\sigma_1 \leqslant 0$.

The admissibility condition can be equivalently expressed with reference to the generic plane elements π_a normal to the lines 'a' passing trough the point P, by considering the normal σ_a and tangential τ_a components of the stress vectors $\mathbf{t}_a$; in this case it imposes that $\sigma_a \leqslant 0$ with τ_a undefined on π_a.

Since the material is not able to resist tensile stresses, it is necessary to allow for the development of inelastic strain $\boldsymbol{\varepsilon}_\mathrm{f}$ (*fracture strain* tensor) superposing to the elastic strain $\boldsymbol{\varepsilon}_\mathrm{e}$.

The fracture strain has the role of transferring those forces deriving from inadmissible tensile stresses to the neighbouring material, in cases where the body has the capacity of achieving equilibrium with the same forces in pure compression.

At any point where the fracture tensor is not zero, contact at the inner of the material is lost on a variety of plane elements. These phenomena are accounted for by assuming that:

- the fracture strain is positive semi-definite (fracture corresponds to a strain state which does not produce contraction of any material element);
- the stress state is negative semi-definite (the stress state cannot suffer tractions);
- on any principal direction where the material is compressed, the relevant coefficient of linear elongation of the fracture strain is zero;

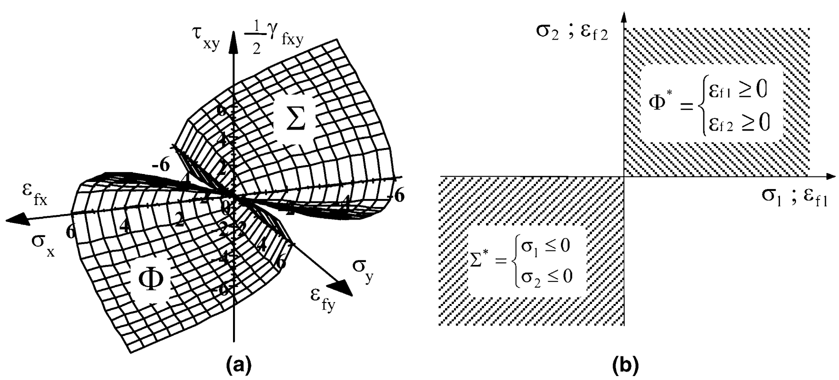

Fig. 1. The stress (Σ) and the fracture (Φ) admissible domains in the spaces of tensor components (a) and of principal components (b).

- when the fracture starts, the displacements relevant to the fracture strain tensor are parallel to the principal direction of the stress tensor corresponding to $\sigma_1 = 0$, that is to say to the compression isostatic lines (in such a manner that no tangential stress rises).

1.2. Standard NRT model

The simplest NRT model assumes linear elasticity under pure compression and development of inelastic strains obeying the Drucker's postulate (Baratta, 1991). Such a model is referred to as *Standard Not Resisting Tension material.*

The Drucker's postulate hypothesis implies the normality rule for the inelastic strain, which is a particularization of the normality law of the plastic flow vector on the plastic surface (Fig. 2); the Drucker's postulate assumption, thus, implies an analogy of the NRT model with elastic–perfectly-plastic associated flow law.

The material should, hence, satisfy the following relations:

$$\begin{cases} \varepsilon_{\mathrm{f}a} \geqslant 0 \\ \sigma_a \leqslant 0 \end{cases} \quad \forall a \in r_p; \quad (\sigma_i < 0) \Rightarrow (\varepsilon_{\mathrm{f}i} = 0); \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\mathrm{e}} + \boldsymbol{\varepsilon}_{\mathrm{f}} = \mathbf{C}\boldsymbol{\sigma} + \boldsymbol{\varepsilon}_{\mathrm{f}} \tag{1}$$

where $\mathbf{C}$ is the elastic tensor, r_p denotes the set of lines passing through the generic point P, a denotes one of these lines.

The Drucker's postulate hypothesis implies the conditions

$$\begin{aligned} &(\boldsymbol{\sigma}' - \boldsymbol{\sigma}) \cdot \boldsymbol{\varepsilon}_{\mathrm{f}} \leqslant 0 \quad \forall \boldsymbol{\sigma}' \in \Sigma \\ &\{\sigma_1 \varepsilon_{\mathrm{f}1} = 0; \sigma_2 \varepsilon_{\mathrm{f}2} = 0\} \Rightarrow \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}_{\mathrm{f}} = 0 \end{aligned} \tag{2}$$

where $\boldsymbol{\sigma}'$ is any admissible stress state other than the effective one $\boldsymbol{\sigma}$.

2. Variational approaches to NRT continua

2.1. Energetic approaches to NRT problems

Analysis of NRT continua proves that the stress, strain and displacement fields obey extremum principles of the basic energy functionals (Del Piero, 1989; Baratta and Voiello, 1996; Bazant and Li, 1995).

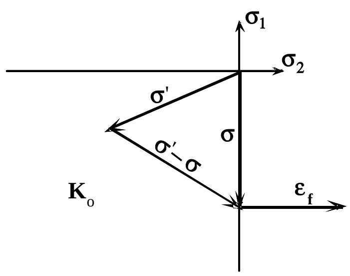

Fig. 2. Admissible stress domain.

The solution of the NRT structural problems can, thus, be referred to the two main variational approaches:

- the minimum principle of the Total Potential Energy functional;
- the minimum principle of the Total Complementary Energy functional.

In the first case the displacements are assumed as independent variables.

Therefore the solution displacement $\mathbf{u}_0$ and fracture strain $\boldsymbol{\varepsilon}_{f0}$ fields are found as the constrained minimum point of the Total Potential Energy functional, under the constraint that the fracture field is positively semi-definite at any point.

The approach based on the minimization of the Total Complementary Energy functional assumes the stress state as independent variable. The complementary approach is widely adopted since the existence and uniqueness of the NRT solution are always ensured in terms of stress, if some conditions on the compatibility of the loads are satisfied.

The stress field $\boldsymbol{\sigma}_0$ can, then, be found as the constrained minimum of the Complementary Energy functional, under the condition that the stress field is in equilibrium with the applied loads and is compressive everywhere.

The solution of both problems can be numerically pursued by means of Operational Research methods (see i.e. Rao, 1978; Zyczkowski, 2002; Cherkaev, 2002; Pedreglal, 2001) suitably operating a discretization of the analysed NRT continuum.

The theorems of Limit Analysis (Baratta, 1991; Como and Grimaldi, 1983; Franciosi, 1980; Khludnev and Kovtunenko, 2000; Bazant, 1989, 1996a,b, 1997) can be specialized to NRT continua after defining the classes of *kinematically sufficient mechanisms* and *statically admissible stress fields*.

2.2. Limit analysis for NRT continua

Denoting by U the set of possible displacement fields, the class of *admissible mechanisms* is defined by the subset U_f of U containing displacement fields $\mathbf{u}_f$ that are directly compatible with fracture strains $\boldsymbol{\varepsilon}_f$ apart from any elastic strain field ("Mechanisms of collapse"), i.e., after introducing the gradient operator ∇

$$\boldsymbol{\varepsilon}_f = \nabla \mathbf{u}_f \geqslant \mathbf{0} \tag{3}$$

$$U_f = \{\mathbf{u}_f \in U:\ \nabla \mathbf{u}_f \geqslant \mathbf{0}\} \tag{4}$$

Kinematically sufficient mechanisms can be defined as fracture admissible mechanisms $\mathbf{u}_f$ such that the energy dissipated by the loads $(\mathbf{p}, \mathbf{F})$ applied on the free surface A_p and the volume Ω is positive; this condition is analytically expressed by the inequality

$$\int_{A_p} \mathbf{p} \cdot \mathbf{u}_f \, \mathrm{d}A + \int_{\Omega} \mathbf{F} \cdot \mathbf{u}_f \, \mathrm{d}V > 0 \tag{5}$$

Since a necessary condition for the existence of the solution is that

$$\int_{A_p} \mathbf{p} \cdot \mathbf{u}_f \, \mathrm{d}A + \int_{\Omega} \mathbf{F} \cdot \mathbf{u}_f \, \mathrm{d}V \leqslant 0 \quad \forall \mathbf{u}_f \in U_f \tag{6}$$

and remembering that the material under examination is unable to dissipate energy (see the second Eq. (2)), one can, thus, enunciate the "Kinematical Theorem" of Limit Analysis for NRT bodies: "if any kinematically sufficient mechanism exists under the applied loads, no solution can exist for the equilibrium of the NRT solid".

On the other side, *statically admissible stress fields* $\boldsymbol{\sigma}$ can be defined as stress tensors equilibrating the applied loads and satisfying admissibility conditions, i.e. $\boldsymbol{\sigma} \in \Sigma$, where Σ is the admissible domain.

Assuming that under the load pattern $(\mathbf{p},\mathbf{F})$ a statically admissible stress field $\boldsymbol{\sigma}$ exists (i.e. $\boldsymbol{\sigma} \leqslant 0$), for any mechanism $\mathbf{u}_\mathrm{f}$, by the Principle of Virtual Work one gets

$$\int_{A_p} \mathbf{p}\cdot\mathbf{u}_\mathrm{f}\,\mathrm{d}A + \int_\Omega \mathbf{F}\cdot\mathbf{u}_\mathrm{f}\,\mathrm{d}V = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_\mathrm{f}\,\mathrm{d}V \leqslant 0 \quad \forall \mathbf{u}_\mathrm{f}\in U_\mathrm{f} \tag{7}$$

One can, thus, enunciate the "Static Theorem" of Limit Analysis for NRT bodies: "if under the applied loads any statically admissible stress field $\boldsymbol{\sigma}$ exists, no kinematically sufficient mechanism exists and the structure cannot collapse".

From the above considerations it follows that the study of the existence and uniqueness of the solution only requires a suitable kind of limit analysis for the structure. Uniqueness of the solution holds for the stress field but not for displacements and strains.

3. Kuhn–Tucker stationarity conditions for minimum energy approaches applied to NRT continua

3.1. Minimum total potential energy approach

Let

$$\mathsf{E}(\mathbf{u},\boldsymbol{\varepsilon}_\mathrm{f}) = \frac{1}{2}\int_\Omega [\nabla\mathbf{u}(\mathbf{x}) - \boldsymbol{\varepsilon}_\mathrm{f}(\mathbf{x})]\cdot\mathbf{C}[\nabla\mathbf{u}(\mathbf{x}) - \varepsilon_\mathrm{f}(\mathbf{x})]\,\mathrm{d}V - \int_{A_p}\mathbf{p}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\,\mathrm{d}A \tag{8}$$

be the Total Potential Energy functional defined on the displacement field $\mathbf{u} = \mathbf{u}(\mathbf{x})$ and the fracture field $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_\mathrm{f}(\mathbf{x})$, where $\mathbf{x}$ denotes the position vector of the current point in the solid, and $\mathbf{C}$ is the tensor of elastic constants of the material.

It is proved that the functional Eq. (8) is minimum in solution, i.e. when the couple of fields $[\mathbf{u}_0(\mathbf{x}),\boldsymbol{\varepsilon}_{\mathrm{f}0}(\mathbf{x})]$ is an *equilibrium* system, over all the fields that satisfy the admissibility conditions for fracture. One can, then, write

$$\mathsf{E}(\mathbf{u}_0,\boldsymbol{\varepsilon}_{\mathrm{f}0}) = \min_{\mathbf{u},\boldsymbol{\varepsilon}_\mathrm{f}} \mathsf{E}(\mathbf{u},\boldsymbol{\varepsilon}_\mathrm{f}) \text{ sub } \begin{cases} J_{1\mathrm{f}}(\mathbf{x}) \geqslant 0 \\ J_{2\mathrm{f}}(\mathbf{x}) \geqslant 0 \end{cases} \quad \forall\mathbf{x}\in\Omega \tag{9}$$

$J_{1\mathrm{f}}(\mathbf{x})$ and $J_{2\mathrm{f}}(\mathbf{x})$ being the linear and quadratic invariants of the fractures field, given by

$$\begin{aligned} J_{1\mathrm{f}}(\mathbf{x}) &= \varepsilon_{\mathrm{f}0x}(\mathbf{x}) + \varepsilon_{\mathrm{f}0y}(\mathbf{x}) \\ J_{2\mathrm{f}}(\mathbf{x}) &= \varepsilon_{\mathrm{f}0x}(\mathbf{x})\varepsilon_{\mathrm{f}0y}(\mathbf{x}) - \left[\frac{1}{2}\gamma_{\mathrm{f}0xy}(\mathbf{x})\right]^2 \end{aligned} \tag{10}$$

It is proved that the solution to the problem Eq. (9) *necessarily yields a negative semi-definite stress field* $\boldsymbol{\sigma}_0(\mathbf{x}) = \mathbf{C}[\nabla\mathbf{u}_0(\mathbf{x}) - \boldsymbol{\varepsilon}_{\mathrm{f}0}(\mathbf{x})]$, *coaxial and orthogonal to the fracture field.*

Let in fact form the Lagrangian functional from the problem equation (9)

$$\mathsf{L}(\mathbf{u},\boldsymbol{\varepsilon}_\mathrm{f},\lambda_1,\lambda_2) = \mathsf{E}(\mathbf{u},\boldsymbol{\varepsilon}_\mathrm{f}) - \int_\Omega \lambda_1(\mathbf{x})J_{1\mathrm{f}}(\mathbf{x})\,\mathrm{d}V - \int_\Omega \lambda_2(\mathbf{x})J_{2\mathrm{f}}(\mathbf{x})\,\mathrm{d}V \tag{11}$$

with $\lambda_1(\mathbf{x})$ and $\lambda_2(\mathbf{x})$ a couple of Lagrange multiplier functions.

By the Kuhn–Tucker optimality conditions, a couple of non negative functions $\lambda_1(\mathbf{x})$ and $\lambda_2(\mathbf{x})$ exists such that *first-order* variations, starting from the solution point $[\mathbf{u}_0(\mathbf{x}),\boldsymbol{\varepsilon}_{\mathrm{f}0}(\mathbf{x})]$, of the Lagrangian functional with respect to free variations of the variables $\mathbf{u}(\mathbf{x})$, $\boldsymbol{\varepsilon}_\mathrm{f}(\mathbf{x})$ are identically zero.

Moreover the complementarity condition holds: $\lambda_1(\mathbf{x})J_{1\mathrm{f}}(\mathbf{x}) = \lambda_2(\mathbf{x})J_{2\mathrm{f}}(\mathbf{x}) = 0\ \forall\mathbf{x}\in\Omega$.

The stationarity condition of the Lagrangian functional with respect to variations $\delta\boldsymbol{\varepsilon}_{\mathrm{f}}$ yields

$$\delta \mathrm{L}_{v}(\mathbf{u}_0, \boldsymbol{\varepsilon}_{\mathrm{f}0}, \lambda_1, \lambda_2) = -\int_{\Omega} \sigma_0(\mathbf{x}) \cdot \delta\boldsymbol{\varepsilon}_{\mathrm{f}}(\mathbf{x})\,\mathrm{d}V - \int_{\Omega} \lambda_1(\mathbf{x})\delta J_{1\mathrm{f}}(\mathbf{x})\,\mathrm{d}V - \int_{\Omega} \lambda_2(\mathbf{x})\delta J_{2\mathrm{f}}(\mathbf{x})\,\mathrm{d}V = 0 \quad \forall \delta\boldsymbol{\varepsilon}_{\mathrm{f}}(\mathbf{x}) \tag{12}$$

In Eq. (12) the variations of the linear and quadratic invariants of the fractures field $\delta J_{1\mathrm{f}}$ and $\delta J_{2\mathrm{f}}$ are expressed by the relations

$$\begin{aligned} &\delta J_{1\mathrm{f}}(\mathbf{x}) = \delta\varepsilon_{\mathrm{f}x}(\mathbf{x}) + \delta\varepsilon_{\mathrm{f}y}(\mathbf{x}) \\ &\delta J_{2\mathrm{f}}(\mathbf{x}) = \varepsilon_{\mathrm{f}0x}(\mathbf{x})\delta\varepsilon_{\mathrm{f}y}(\mathbf{x}) + \varepsilon_{\mathrm{f}0y}(\mathbf{x})\delta\varepsilon_{\mathrm{f}x}(\mathbf{x}) - \frac{1}{2}\gamma_{\mathrm{f}0xy}(\mathbf{x})\delta\gamma_{xy}(\mathbf{x}) \end{aligned} \tag{13}$$

Moreover the scalar product in the first integral in Eq. (12) can be developed as follows:

$$\begin{aligned} &\int_{\Omega} \{[\sigma_{0x}(\mathbf{x}) + \lambda_1(\mathbf{x}) + \lambda_2(\mathbf{x})\varepsilon_{\mathrm{f}0y}(\mathbf{x})]\delta\varepsilon_{\mathrm{f}x}\}\,\mathrm{d}V + \int_{\Omega} \{[\sigma_{0y}(\mathbf{x}) + \lambda_1(\mathbf{x}) + \lambda_2(\mathbf{x})\varepsilon_{\mathrm{f}0x}]\delta\varepsilon_{\mathrm{f}y}(\mathbf{x})\}\,\mathrm{d}V \\ &\quad + \int_{\Omega} \left\{ \left[\tau_{0xy}(\mathbf{x}) - \frac{1}{2}\lambda_2(\mathbf{x})\gamma_{\mathrm{f}0xy}(\mathbf{x}) \right] \delta\gamma_{xy}(\mathbf{x}) \right\} \mathrm{d}V = 0 \quad \forall(\delta\varepsilon_{\mathrm{f}x}, \delta\varepsilon_{\mathrm{f}y}, \delta\gamma_{xy}) \end{aligned} \tag{14}$$

which implies

$$\begin{cases} \sigma_{0x}(\mathbf{x}) = -\lambda_1(\mathbf{x}) - \lambda_2(\mathbf{x})\varepsilon_{\mathrm{f}0y}(\mathbf{x}) \\ \sigma_{0y}(\mathbf{x}) = -\lambda_1(\mathbf{x}) - \lambda_2(\mathbf{x})\varepsilon_{\mathrm{f}0x}(\mathbf{x}) \\ \tau_{0xy}(\mathbf{x}) = \frac{1}{2}\lambda_2(\mathbf{x})\gamma_{\mathrm{f}0xy}(\mathbf{x}) \end{cases} \tag{15}$$

One can, thus, easily calculate the stress invariants $I_1(\mathbf{x})$ and $I_2(\mathbf{x})$

$$\begin{aligned} &I_1(\mathbf{x}) = \sigma_{0x}(\mathbf{x}) + \sigma_{0y}(\mathbf{x}) = -2\lambda_1(\mathbf{x}) - \lambda_2(\mathbf{x})J_{1\mathrm{f}}(\mathbf{x}) \leqslant 0 \\ &I_2(\mathbf{x}) = \sigma_{0x}(\mathbf{x})\sigma_{0y}(\mathbf{x}) - [\tau_{0xy}(\mathbf{x})]^2 = [\lambda_1(\mathbf{x})]^2 + \lambda_1(\mathbf{x})\lambda_2(\mathbf{x})J_{1\mathrm{f}}(\mathbf{x}) + [\lambda_2(\mathbf{x})]^2 J_{2\mathrm{f}}(\mathbf{x}) \geqslant 0 \end{aligned} \tag{16}$$

whence one can conclude that

(i) "The stress tensor in every element is negative semi-definite in solution"Consider now the *fracture work*

$$\begin{aligned} L_{\mathrm{f}}(\mathbf{x}) &= \boldsymbol{\sigma}_0(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{\mathrm{f}0}(\mathbf{x}) = \sigma_{0x}(\mathbf{x})\varepsilon_{\mathrm{f}0x}(\mathbf{x}) + \sigma_{0y}(\mathbf{x})\varepsilon_{\mathrm{f}0y}(\mathbf{x}) + \tau_{0xy}(\mathbf{x})\gamma_{\mathrm{f}0xy}(\mathbf{x}) \\ &= -[\lambda_1(\mathbf{x}) + \lambda_2(\mathbf{x})\varepsilon_{\mathrm{f}0y}(\mathbf{x})]\varepsilon_{\mathrm{f}0x}(\mathbf{x}) - [\lambda_1(\mathbf{x})\lambda_2(\mathbf{x})\varepsilon_{\mathrm{f}0x}(\mathbf{x})]\varepsilon_{\mathrm{f}0y}(\mathbf{x}) + \frac{1}{2}\lambda_2(\mathbf{x})[\gamma_{\mathrm{f}0xy}(\mathbf{x})]^2 \end{aligned} \tag{17}$$

whence, after some algebra

$$L_{\mathrm{f}}(\mathbf{x}) = -\lambda_1(\mathbf{x})J_{1\mathrm{f}}(\mathbf{x}) - 2\lambda_2(\mathbf{x})J_{2\mathrm{f}}(\mathbf{x}) = 0 \tag{18}$$

and one can conclude that

(ii) "The fracture work in every element is null in solution".Let us now consider the *orientation* of the stress tensor in solution.The eigenvectors $\boldsymbol{\alpha}_i(\mathbf{x})$ and $\boldsymbol{\beta}_i(\mathbf{x})$ (for $i = 1, 2$) of, respectively, the stress and fracture-strain tensors obey the equations

$$\text{Stress:} \quad \begin{aligned} &\begin{cases} [\sigma_{0x}(\mathbf{x}) - \sigma_{01}(\mathbf{x})]\alpha_{1x}(\mathbf{x}) + \tau_{0xy}(\mathbf{x})\alpha_{1y}(\mathbf{x}) = 0 \\ \tau_{0xy}(\mathbf{x})\alpha_{1x}(\mathbf{x}) + [\sigma_{0y}(\mathbf{x}) - \sigma_{01}(\mathbf{x})]\alpha_{1y}(\mathbf{x}) = 0 \end{cases} \\ &\begin{cases} [\sigma_{0x}(\mathbf{x}) - \sigma_{02}(\mathbf{x})]\alpha_{2x}(\mathbf{x}) + \tau_{0xy}(\mathbf{x})\alpha_{2y}(\mathbf{x}) = 0 \\ \tau_{0xy}(\mathbf{x})\alpha_{2x}(\mathbf{x}) + [\sigma_{0y}(\mathbf{x}) - \sigma_{02}(\mathbf{x})]\alpha_{2y}(\mathbf{x}) = 0 \end{cases} \end{aligned} \tag{19}$$

$$\text{Fractures:}\quad \begin{cases} [\varepsilon_{f0x}(\mathbf{x}) - \varepsilon_{f01}(\mathbf{x})]\beta_{1x}(\mathbf{x}) + \frac{1}{2}\gamma_{f0xy}(\mathbf{x})\beta_{1y}(\mathbf{x}) = 0 \\ \frac{1}{2}\gamma_{f0xy}(\mathbf{x})\beta_{1x}(\mathbf{x}) + [\varepsilon_{f0y}(\mathbf{x}) - \varepsilon_{f01}(\mathbf{x})]\beta_{1y}(\mathbf{x}) = 0 \end{cases} \quad \begin{cases} [\varepsilon_{f0x}(\mathbf{x}) - \varepsilon_{f02}(\mathbf{x})]\beta_{2x}(\mathbf{x}) + \frac{1}{2}\gamma_{f0xy}(\mathbf{x})\beta_{2y}(\mathbf{x}) = 0 \\ \frac{1}{2}\gamma_{f0xy}(\mathbf{x})\beta_{2x}(\mathbf{x}) + [\varepsilon_{f0y}(\mathbf{x}) - \varepsilon_{f02}(\mathbf{x})]\beta_{2y}(\mathbf{x}) = 0 \end{cases} \tag{20}$$

where $\sigma_{0i}(\mathbf{x})$ and $\varepsilon_{f0i}(\mathbf{x})$ with $i = 1, 2$ are the eigenvalues of, respectively, the stress and fracture-strain tensors.By means of Eq. (15) it can be easily checked that

$$\begin{cases} \sigma_{01}(\mathbf{x}) = -[\lambda_1(\mathbf{x}) + \lambda_2(\mathbf{x})\varepsilon_{f02}(\mathbf{x})] \\ \sigma_{02}(\mathbf{x}) = -[\lambda_1(\mathbf{x}) + \lambda_2(\mathbf{x})\varepsilon_{f01}(\mathbf{x})] \end{cases} \tag{21}$$

Substitution into the first equation of the first set of Eq. (19) for $\boldsymbol{\alpha}_1(\mathbf{x})$, yields, remembering Eq. (15)

$$\lambda_2(\mathbf{x})[\varepsilon_{f0y}(\mathbf{x}) - \varepsilon_{f02}(\mathbf{x})]\alpha_{1x}(\mathbf{x}) - \frac{1}{2}\lambda_2(\mathbf{x})\gamma_{f0xy}(\mathbf{x})\alpha_{1y}(\mathbf{x}) = 0 \tag{22}$$

whence, by comparison with the second equation of the second set in Eq. (20) for $\boldsymbol{\beta}_2(\mathbf{x})$

$$\frac{1}{2}\gamma_{f0xy}(\mathbf{x})\beta_{2x}(\mathbf{x}) - [\varepsilon_{f0y}(\mathbf{x}) - \varepsilon_{f02}(\mathbf{x})]\beta_{2y}(\mathbf{x}) = 0 \tag{23}$$

one gets

$$\beta_{2x}(\mathbf{x}) = -\alpha_{1y}(\mathbf{x}); \quad \beta_{2y}(\mathbf{x}) = \alpha_{1x}(\mathbf{x}) \tag{24}$$

Eq. (24) states that $\boldsymbol{\alpha}_1(\mathbf{x})$ is orthogonal to $\boldsymbol{\beta}_2(\mathbf{x})$ and consequently parallel to $\boldsymbol{\beta}_1$ $(\mathbf{x})$.Therefore, one concludes that:

(iii) "The stress tensor in every element is coaxial to the fracture strain tensor in solution".

The above demonstrated statements (i), (ii) and (iii) can be analytically expressed by the conditions:

$$\begin{aligned} &\sigma_0(\mathbf{x}) \leqslant \mathbf{0} \\ &\sigma_0(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{f0}(\mathbf{x}) = 0 \\ &\sigma_0(\mathbf{x}) \text{ coaxial } \boldsymbol{\varepsilon}_{f0}(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega \end{aligned} \tag{25}$$

From the stationarity condition of the Lagrangian functional given in Eq. (11) with respect to variations $\delta\mathbf{u}(\mathbf{x})$, one gets

$$\delta \mathsf{L}_{\mathbf{u}}(\mathbf{u}_0, \varepsilon_{f0}, \lambda_1, \lambda_2) = \int_\Omega \boldsymbol{\sigma}_0(\mathbf{x}) \cdot [\nabla \delta\mathbf{u}(\mathbf{x}) - \varepsilon_{f0}(\mathbf{x})]\,\mathrm{d}V - \int_{A_p} \mathbf{p}(\mathbf{x}) \cdot \delta\mathbf{u}(\mathbf{x})\,\mathrm{d}A = 0 \quad \forall \delta\mathbf{u}(\mathbf{x}) \tag{26}$$

which, remembering that $\sigma_0(\mathbf{x}) \cdot \varepsilon_{f0}(\mathbf{x}) = 0$ $\forall \mathbf{x} \in \Omega$, as already found in above, is simply the variational condition for equilibrium of the stress field in solution (the Principle of Virtual Work).

3.2. Minimum total complementary energy approach

Let $\mathbf{r}(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x})\boldsymbol{\alpha}_n(\mathbf{x})$ be the reactive surface traction on the constrained surface A_d and $\mathbf{u}_d(\mathbf{x})$ the imposed displacements. On the stress field $\boldsymbol{\sigma}(\mathbf{x})$, let define

$$\mathsf{U}(\boldsymbol{\sigma}) = \frac{1}{2}\int_\Omega \boldsymbol{\sigma}(x) \cdot \mathbf{D}\boldsymbol{\sigma}(\mathbf{x})\,\mathrm{d}V - \int_{A_d} \mathbf{r}(\mathbf{x}) \cdot \mathbf{u}_d(\mathbf{x})\,\mathrm{d}A \tag{27}$$

as the Complementary Energy functional, with $\mathbf{D}$ the compliance tensor.

It is proved that the functional in Eq. (27) is minimum in solution, i.e. when the field $\boldsymbol{\sigma}_0(\mathbf{x})$ is a *admissible* system (that means a fracture positive semi-definite strain field exists such that after addition to the elastic strains a compatible strain field is produced), over all the fields that are negative semi-definite and satisfy the equilibrium. In other words

$$\mathsf{U}(\boldsymbol{\sigma}_0) = \min_{\boldsymbol{\sigma}} \mathsf{U}(\boldsymbol{\sigma}) \text{ sub} \begin{cases} \begin{cases} I_1(\mathbf{x}) \leqslant 0 \\ I_2(\mathbf{x}) \geqslant 0 & \forall \mathbf{x} \in \Omega \\ \mathbf{Div}\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \end{cases} \\ \boldsymbol{\sigma}(\mathbf{x})\boldsymbol{\alpha}_n(\mathbf{x}) = \mathbf{p}(\mathbf{x}) \quad \forall \mathbf{x} \in A_d \end{cases} \tag{28}$$

$I_1(\mathbf{x})$ and $I_2(\mathbf{x})$ being the linear and quadratic invariants of the stress field.

It is proved that the solution to the problem Eq. (28) *necessarily yields a positive semi-definite inelastic field* $\boldsymbol{\varepsilon}_{f0}(\mathbf{x}) = \nabla\mathbf{u}(\mathbf{x}) - \mathbf{D}\boldsymbol{\sigma}_0(\mathbf{x})$, *coaxial and orthogonal to the stress field.*

Let in fact form the Lagrangian functional from the problem Eq. (28)

$$\mathsf{L}(\boldsymbol{\sigma}|\lambda, \mu, \omega_1, \omega_2) = \mathsf{U}(\boldsymbol{\sigma}) + \int_\Omega \lambda(\mathbf{x}) \cdot [\mathbf{Div}\boldsymbol{\sigma}(\mathbf{x}) - \mathbf{F}(\mathbf{x})]\,\mathrm{d}V + \int_{A_d} \mu(\mathbf{x}) \cdot [\mathbf{p}(\mathbf{x}) - \boldsymbol{\sigma}(\mathbf{x})\boldsymbol{\alpha}_n(\mathbf{x})]\,\mathrm{d}A$$
$$+ \int_\Omega \omega_1(\mathbf{x}) I_1(\mathbf{x})\,\mathrm{d}V - \int_\Omega \omega_2(\mathbf{x}) I_2(\mathbf{x})\,\mathrm{d}V \tag{29}$$

with $\boldsymbol{\lambda}(\mathbf{x})$ and $\boldsymbol{\mu}(\mathbf{x})$ a couple of 3-dimensional vector functions of Lagrange multipliers, and $\omega_1(\mathbf{x})$, $\omega_2(\mathbf{x})$ a couple of scalar functions.

By the Kuhn–Tucker optimality conditions, $\boldsymbol{\lambda}(\mathbf{x})$ and $\boldsymbol{\mu}(\mathbf{x})$ and a couple of non negative functions $\omega_1(\mathbf{x})$ and $\omega_2(\mathbf{x})$ exist such that first-order variations, starting from the solution point $\boldsymbol{\sigma}_0(\mathbf{x})$ of the Lagrangian functional with respect to free variations of the variable $\boldsymbol{\sigma}(\mathbf{x})$, are identically zero. Moreover the complementarity condition yields: $\omega_1(\mathbf{x})I_1(\mathbf{x}) = \omega_2(\mathbf{x})I_2(\mathbf{x}) = 0\ \forall \mathbf{x} \in \Omega$.

The stationarity condition of the Lagrangian with respect to variations $\delta\boldsymbol{\sigma}$ yields

$$\delta\mathsf{L}_\sigma(\boldsymbol{\sigma}_0|\lambda, \mu, \omega_1, \omega_2) = \int_\Omega \mathbf{D}\boldsymbol{\sigma}_0(\mathbf{x}) \cdot \delta\boldsymbol{\sigma}(\mathbf{x})\,\mathrm{d}V - \int_{A_d} [\delta\boldsymbol{\sigma}(\mathbf{x})\boldsymbol{\alpha}_n(\mathbf{x})] \cdot \mathbf{u}_d(\mathbf{x})\,\mathrm{d}A - \int_{A_p} [\delta\boldsymbol{\sigma}(\mathbf{x})\boldsymbol{\alpha}_n(\mathbf{x})] \cdot \mu(\mathbf{x})\,\mathrm{d}A$$
$$+ \int_\Omega \lambda(\mathbf{x}) \cdot [\mathbf{Div}\delta\boldsymbol{\sigma}(\mathbf{x})]\,\mathrm{d}V + \int_\Omega \omega_1(\mathbf{x})\delta I_1(\mathbf{x})\,\mathrm{d}V$$
$$- \int_\Omega \omega_2(\mathbf{x})\delta I_2(\mathbf{x})\,\mathrm{d}V = 0 \quad \forall \delta\boldsymbol{\sigma}(\mathbf{x}) \tag{30}$$

whence, remembering that $\lambda(\mathbf{x}) \cdot [\mathbf{Div}\delta\boldsymbol{\sigma}(\mathbf{x})] = \mathrm{div}[\delta\boldsymbol{\sigma}(\mathbf{x})\lambda(\mathbf{x})] - \delta\boldsymbol{\sigma}(\mathbf{x}) \cdot \mathbf{Grad}\,\lambda(\mathbf{x})$ and applying the Gauss's theorem one obtains for the fourth integral in Eq. (30)

$$\int_\Omega \lambda(\mathbf{x}) \cdot [\mathbf{Div}\delta\boldsymbol{\sigma}(\mathbf{x})]\,\mathrm{d}V = \int_\Omega \mathrm{div}[\delta\boldsymbol{\sigma}(\mathbf{x})\lambda(\mathbf{x})]\,\mathrm{d}V - \int_\Omega \delta\boldsymbol{\sigma}(\mathbf{x}) \cdot \mathbf{Grad}\,\lambda(\mathbf{x})\,\mathrm{d}V$$
$$= \int_A [\delta\boldsymbol{\sigma}(\mathbf{x})\lambda(\mathbf{x})] \cdot \boldsymbol{\alpha}_n(\mathbf{x})\,\mathrm{d}A - \int_\Omega \delta\boldsymbol{\sigma}(\mathbf{x}) \cdot \mathbf{Grad}\,\lambda(\mathbf{x})\,\mathrm{d}V \tag{31}$$

with $A = A_p \cup A_d$.

Introducing this result in Eq. (30) one gets

$$\delta\mathsf{L}_\sigma(\boldsymbol{\sigma}_0|\lambda, \mu, \omega_1, \omega_2) = \int_\Omega \delta\boldsymbol{\sigma}(\mathbf{x}) \cdot [\mathbf{D}\boldsymbol{\sigma}_0(\mathbf{x}) - \mathbf{Grad}\,\lambda(\mathbf{x})]\,\mathrm{d}V - \int_{A_d} [\delta\boldsymbol{\sigma}(\mathbf{x})\alpha_n(\mathbf{x})] \cdot \mathbf{u}_d(\mathbf{x})\,\mathrm{d}A$$
$$- \int_{A_p} [\delta\boldsymbol{\sigma}(\mathbf{x})\alpha_n(\mathbf{x})] \cdot \mu(\mathbf{x})\,\mathrm{d}A + \int_A [\delta\boldsymbol{\sigma}(\mathbf{x})\lambda(\mathbf{x})] \cdot \alpha_n(\mathbf{x})\,\mathrm{d}A + \int_\Omega \omega_1(\mathbf{x})\delta I_1(\mathbf{x})\,\mathrm{d}V$$
$$- \int_\Omega \omega_2(\mathbf{x})\delta I_2(\mathbf{x})\,\mathrm{d}V = 0 \quad \forall \delta\boldsymbol{\sigma}(\mathbf{x}) \tag{32}$$

The variational Eq. (32) can be solved by identifying the vector field $\boldsymbol{\lambda}(\mathbf{x})$ in the displacement field $\mathbf{u}_0(\mathbf{x})$ of the solid in solution. This implies that $\boldsymbol{\lambda}(\mathbf{x})$ coincides with imposed displacements $\mathbf{u_d}(\mathbf{x})$ on A_d, and that $\mathbf{Grad}\,\boldsymbol{\lambda}(\mathbf{x}) = \boldsymbol{\varepsilon}_0(\mathbf{x})$, the strain field in solution. The vector field $\boldsymbol{\mu}(\mathbf{x})$, in turn, is assumed to coincide with the free displacements on A_p, in such a way that

$$\begin{aligned}&\mathbf{Grad}\,\boldsymbol{\lambda}(\mathbf{x}) = \boldsymbol{\varepsilon}_0(\mathbf{x}); \quad \boldsymbol{\lambda}(\mathbf{x}) \equiv \mathbf{u_o}(\mathbf{x}) \text{ in } \Omega; \quad \boldsymbol{\lambda}(\mathbf{x}) \equiv \mathbf{u_d}(\mathbf{x}) \text{ on } A_d \\ &\boldsymbol{\mu}(\mathbf{x}) \equiv \mathbf{u_o}(\mathbf{x}) \equiv \boldsymbol{\lambda}(\mathbf{x}) \text{ on } A_d\end{aligned} \tag{33}$$

As a consequence $\mathbf{D}\boldsymbol{\sigma}_0(\mathbf{x}) - \mathbf{Grad}\,\boldsymbol{\lambda}(\mathbf{x}) = -\boldsymbol{\varepsilon}_{\mathrm{f}0}(\mathbf{x})$ and after recognizing that

$$\int_{A_d} [\delta\boldsymbol{\sigma}(\mathbf{x})\boldsymbol{\alpha}_n(\mathbf{x})] \cdot \mathbf{u}_d(\mathbf{x})\,\mathrm{d}A + \int_{A_p} [\delta\boldsymbol{\sigma}(\mathbf{x})\boldsymbol{\alpha}_n(\mathbf{x})] \cdot \mu(\mathbf{x})\,\mathrm{d}A = \int_A [\delta\boldsymbol{\sigma}(\mathbf{x})\boldsymbol{\lambda}(\mathbf{x})] \cdot \boldsymbol{\alpha}_n(\mathbf{x})\,\mathrm{d}A \tag{34}$$

Eq. (32) yields

$$\delta\mathsf{L}_\sigma(\boldsymbol{\sigma}_0|\lambda,\mu,\omega_1,\omega_2) = -\int_\Omega \delta\boldsymbol{\sigma}(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{\mathrm{f}0}(\mathbf{x}) + \int_\Omega \omega_1(\mathbf{x})\delta I_1(\mathbf{x})\,\mathrm{d}V - \int_\Omega \omega_2(\mathbf{x})\delta I_2(\mathbf{x})\,\mathrm{d}V = 0 \quad \forall\delta\boldsymbol{\sigma}(\mathbf{x}) \tag{35}$$

Remembering that

$$\begin{aligned}&\delta I_1(\mathbf{x}) = \delta\sigma_x(\mathbf{x}) + \delta\sigma_y(\mathbf{x}) \\ &\delta I_2(\mathbf{x}) = \sigma_{0x}(\mathbf{x})\delta\sigma_y(\mathbf{x}) + \sigma_{0y}(\mathbf{x})\delta\sigma_x(\mathbf{x}) - 2\tau_{0xy}(\mathbf{x})\delta\tau_{xy}(\mathbf{x})\end{aligned} \tag{36}$$

and developing the scalar product in the first integral in Eq. (35)

$$\begin{aligned}&\int_\Omega \{[-\varepsilon_{\mathrm{f}0x}(\mathbf{x}) + \omega_1(\mathbf{x}) - \omega_2(\mathbf{x})\sigma_{0y}(\mathbf{x})]\delta\sigma_x\}\,\mathrm{d}V + \int_\Omega \{[-\varepsilon_{\mathrm{f}0y}(\mathbf{x}) + \omega_1(\mathbf{x}) - \omega_2(\mathbf{x})\sigma_{0x}]\delta\sigma_y(\mathbf{x})\}\,\mathrm{d}V \\ &\quad + \int_\Omega \{[-\gamma_{\mathrm{f}0xy}(\mathbf{x}) + 2\omega_2(\mathbf{x})\tau_{0xy}(\mathbf{x})]\delta\tau_{xy}(\mathbf{x})\}\,\mathrm{d}V = 0 \quad \forall(\delta\sigma_x,\delta\sigma_y,\delta\tau_{xy})\end{aligned} \tag{37}$$

which implies

$$\begin{cases}\varepsilon_{\mathrm{f}0x}(\mathbf{x}) = \omega_1(\mathbf{x}) - \omega_2(\mathbf{x})\sigma_{0y}(\mathbf{x}) \\ \varepsilon_{\mathrm{f}0y}(\mathbf{x}) = \omega_1(\mathbf{x}) - \omega_2(\mathbf{x})\sigma_{0x}(\mathbf{x}) \\ \gamma_{\mathrm{f}0xy}(\mathbf{x}) = 2\omega_2(\mathbf{x})\tau_{0xy}(\mathbf{x})\end{cases} \tag{38}$$

The fracture–strain invariants are easily calculated

$$\begin{aligned}&J_{1\mathrm{f}}(\mathbf{x}) = \varepsilon_{\mathrm{f}0x}(\mathbf{x}) + \varepsilon_{\mathrm{f}0y} = 2\omega_1(\mathbf{x}) - \omega_2(\mathbf{x})I_1(\mathbf{x}) \geqslant 0 \\ &J_{2\mathrm{f}}(\mathbf{x}) = \varepsilon_{\mathrm{f}0x}(\mathbf{x})\varepsilon_{\mathrm{f}0y}(\mathbf{x}) - \left[\frac{1}{2}\gamma_{\mathrm{f}0xy}(\mathbf{x})\right]^2 = \omega_1^2(\mathbf{x}) - \omega_1(\mathbf{x})\omega_2(\mathbf{x})I_1(\mathbf{x}) + \omega_2^2(\mathbf{x})I_2(\mathbf{x}) \geqslant 0\end{aligned} \tag{39}$$

whence one can conclude that

(i) "The fracture-strain tensor in every element is positive semi-definite in solution".Consider now the *fracture work*

$$\begin{aligned}L_\mathrm{f}(\mathbf{x}) &= \boldsymbol{\sigma}_0 \cdot \boldsymbol{\varepsilon}_{\mathrm{f}0} = \sigma_{0x}(\mathbf{x})\varepsilon_{\mathrm{f}0x}(\mathbf{x}) + \sigma_{0y}(\mathbf{x})\varepsilon_{\mathrm{f}0y}(\mathbf{x}) + \tau_{0xy}(\mathbf{x})\gamma_{\mathrm{f}0xy}(\mathbf{x}) \\ &= [\omega_1(\mathbf{x}) - \omega_2(\mathbf{x})\sigma_{0y}(\mathbf{x})]\sigma_{0x}(\mathbf{x}) + [\omega_1(\mathbf{x}) - \omega_2(\mathbf{x})\sigma_{0x}(\mathbf{x})]\sigma_{0y}(\mathbf{x}) + 2\omega_2(\mathbf{x})[\tau_{0xy}(\mathbf{x})]^2\end{aligned} \tag{40}$$

whence, after some algebra

$$L_\mathrm{f}(\mathbf{x}) = \omega_1(\mathbf{x})I_1(\mathbf{x}) - 2\omega_2(\mathbf{x})I_2(\mathbf{x}) = 0 \tag{41}$$

and one can conclude that

(ii) "The fracture work in every element is null in solution".Let us now consider the *orientation* of the fracture-strain tensor in solution. Since Eq. (38) are quite analogous to Eq. (15) it is easy to follow a rationale similar to Eqs. (19)–(24) to conclude that:

(iii) "The fracture-strain tensor in every element is coaxial to the stress tensor in solution".

The above demonstrated statements (i), (ii) and (iii) can be analytically expressed by the conditions:

$$\begin{aligned} &\boldsymbol{\varepsilon}_{f0}(\mathbf{x}) \geqslant \mathbf{0} \\ &\boldsymbol{\sigma}_0(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{f0}(\mathbf{x}) = 0 \\ &\boldsymbol{\sigma}_0(\mathbf{x}) \text{ coaxial } \boldsymbol{\varepsilon}_{f0}(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega \end{aligned} \tag{42}$$

4. An application: The NRT elastic semi-plane

4.1. The NRT elastic semi-plane

The above presented theory can be applied for analysing the stress distribution in a NRT elastic semi-space under the action of a distributed load; this condition may be intended to reproduce the stress situation induced in the soil by a foundation structure, since the soil behaviour is characterized by a very low resistance to tensile stresses.

In the following, the Complementary Energy approach is adopted for an approximate solution, since, as above mentioned, the existence and uniqueness of the stress solution for a NRT material are always ensured if the loads obey compatibility conditions.

Let, then, consider the case of the indefinite semi-plane shown in Fig. 3, subject to in-plane volume and superficial loads acting in the two directions x and y; the semi-plane stress field obeys indefinite and boundary equilibrium conditions, and admissibility conditions, which depend on the material characterization. The semi-plane is described by $y < 0$.

Since one can assume an arbitrary expression for the surface loads, one follows a *distributional* approach: α-type load distributions are adopted, whose sequence converges to the Dirac distribution.

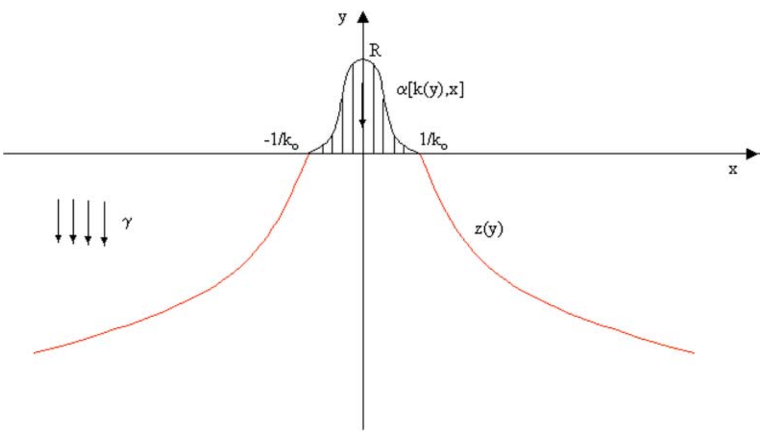

Fig. 3. The NRT elastic semi-plane with the load pattern.

Therefore, with reference to Fig. 3, one assumes on the semi-plane limit line $y = 0$ a vertical distributed load $q(x)$

$$q(x) = R\alpha[k(0), x] \text{ with } \alpha[k(y), x] = \begin{cases} 0 & |x| \geqslant 1/k(y) \\ k(y)\rho e^{-\frac{1}{1-[k(y)x]^2}} & |x| \leqslant 1/k(y) \end{cases}, \quad \rho \int_{-1}^{1} e^{-\frac{1}{1-x^2}} dx = 1 \tag{43}$$

where R is the load resultant ($R < 0$) and $\alpha[k(y), x]$ is a continuous function, indefinitely derivable and such that $\text{supp}\{\alpha[k(y), x]\} = [-1/k(y), 1/k(y)]$.

By varying the function $k(y)$ and, therefore, the support of the distributed surface load $q(x)$, one can build up a class of functions $\alpha[k(y), x]$, whose limit for $k(y) \to \infty$ coincides with the impulsive Dirac's function δ. From Fig. 3 one can observe that, for $y < 0$, the relation between the curve $z(y)$ modelling the process of stress diffusion in the semi-plane and $k(y)$ is $z(y) = 1/k(y)$.

The stress field described in the NRT elastic semi-plane is described by the components $\sigma_x(x, y)$, $\sigma_y(x, y)$ and $\tau_{xy}(x, y)$, which should obey static equilibrium and admissibility conditions.

In the specific case, one considers only the self-weight forces related to the specific weight for volume unit γ and the distributed load $q(x)$ given in Eq. (43).

Therefore equilibrium conditions can be written as follows:

$$\begin{cases} \dfrac{\partial \sigma_x(x, y)}{\partial x} + \dfrac{\partial \tau_{xy}(x, y)}{\partial y} = 0 \\ \dfrac{\partial \tau_{xy}(x, y)}{\partial x} + \dfrac{\partial \sigma_y(x, y)}{\partial y} - \gamma = 0 \end{cases} \tag{44}$$

$$\begin{cases} \tau_{xy}(x, 0) = 0 \\ \sigma_y(x, 0) = q(x) = R\alpha[k(0), x] \end{cases} \tag{45}$$

Moreover, the symmetry of the treated problem is accounted for by introducing the relation

$$\tau_{xy}(0, y) = 0 \tag{46}$$

By means of the symmetry with respect to the y-axis and marking by $H(\cdot)$ the Heaviside function, the general expression of the $\sigma_y(x, y)$ stress component in the semi-plane can be written in the form

$$\sigma_y(x, y) = H[1 - k(y)x]Rk(y)\rho e^{-\frac{1}{1-[k(y)x]^2}} + \gamma y \tag{47}$$

Because of the properties characterizing the Dirac's distribution, the following relations hold:

$$H'(t) = \delta(t), \quad f(t)\delta(t) = f(0)\delta(t) \tag{48}$$

the stress field obeying the equilibrium conditions in the NRT elastic semi-plane is expressed by Eq. (47) and the relations that can be consequently derived (after integration) by means of Eqs. (44) and (45), i.e.

$$\tau_{xy}(x, y) = -H[1 - k(y)x]Rk'(y)\rho x e^{-\frac{1}{1-[k(y)x]^2}} + \tau_0(y) \tag{49}$$

$$\begin{aligned} \sigma_x(x, y) = & R\rho \left[\frac{k''(y)}{k(y)^2} - \frac{2k'(y)^2}{k(y)^3} \right] \cdot \{H[k(y)x - 1]\Pi(1) + H[1 - k(y)x]\Pi[k(y)x]\} \\ & + H[1 - k(y)x]R\rho \frac{k'(y)^2}{k(y)} x^2 e^{-\frac{1}{1-[k(y)x]^2}} + \sigma_0(y), \quad \Pi[k(y)x] = \int_0^{k(y)x} w e^{-\frac{1}{1-w^2}} dw \end{aligned} \tag{50}$$

with the symbols $(\cdot)'$ and $(\cdot)''$ indicating first and second order derivatives, and $\tau_0(y)$ and $\sigma_0(y)$ denoting unknown functions of y coming out from integration.

In details, $\tau_0(y)$ and $\sigma_0(y)$ represent the stress distribution at $x = 0$; from Eq. (46) one gets

$$\tau_{xy}(0, y) = \tau_0(y) = 0 \quad \forall y \tag{51}$$

while the function $\sigma_0(y)$ is assumed to be composed by two terms according to the relation

$$\sigma_x(0,y) = \sigma_0(y) = \sigma_0^p(y) + \sigma_0^q(y) = v\gamma y - R\rho\left[\frac{k''(y)}{k(y)^2} - \frac{2k'(y)^2}{k(y)^3}\right]\Pi(1) \tag{52}$$

with v the Poisson's coefficient.

From Eqs. (47), (49) and (50), expressing the equilibrated stress components, one can observe that no discontinuity in the stress field can be detected in correspondence of the curve $z(y)$. One can, thus, build up equilibrated stress fields by specifying the function $k(y)$. The internal equilibrium is ensured by the continuity and double derivability of such function, while the boundary equilibrium Eq. (45), determines an initial condition on the $k(y)$ first derivative, that is $k'(0) = 0$.

Since for y $\to -\infty$, $k(y) \to 0$, that is to say that it is expected that the influence of the local load decreases and the solution almost depends on the only vertical volume forces, one assumes an expression of the $k(y)$ function of the type

$$k(y) = k_0(y)e^{-\beta y^2} = \left(\sum_{i=0}^{n} k_i y^i\right)e^{-\beta y^2} \tag{53}$$

where β is a positive constant to be determined and $k_0(y)$ is a function of y to be given in such a manner that the initial value of $k(y)$ coincides with the one of $k_0(y)$, i.e. $k_0(0) = k(0) = k_0$, which represents the load trace on the boundary surface $y = 0$. Finally k_i, for $i = 1 \dots n$, denote n unknown coefficients.

The expression of $k(y)$ in Eq. (53) is obviously continuous, characterized by continuous derivatives (C^∞ class), and satisfies $k'(0) = 0$, provided that $k_1 = 0$ if it is the case.

Dealing with an NRT elastic plane, the stress field characterizing the problem solution should comply also with material admissibility conditions.

Therefore, the stress components obtained by imposing equilibrium conditions, should also satisfy the conditions

$$\sigma_x(x,y) \leqslant 0, \quad \sigma_y(x,y) \leqslant 0, \quad \sigma_x(x,y)\sigma_y(x,y) \geqslant \tau_{xy}^2(x,y) \tag{54}$$

which ensure pure compression on any other surface element in any point of the semi-plane.

As above stated, by the complementary energy approach the solution stress field $\boldsymbol{\sigma}_0$ should comply with equilibrium and admissibility conditions. In details, one should search for the problem solution in the class of equilibrated solutions, which are defined by stress fields expressed by Eqs. (47), (49) and (50) and the related Eqs. (51) and (52); at the same time, the solution should satisfy the inequalities in Eq. (54).

Therefore the problem can be set as follows:

$$\mathsf{U}[\boldsymbol{\sigma}_0(x,y)] = \min_{\sigma(x,y)} \mathsf{U}[\boldsymbol{\sigma}(x,y)] = \min_{z(y)} \mathsf{U}[z(y)] \ \text{sub}\begin{cases} \sigma_x(x,y) \leqslant 0 \\ \sigma_y(x,y) \leqslant 0 \\ \sigma_x(x,y)\sigma_y(x,y) \geqslant \tau_{xy}^2(x,y) \end{cases} \quad \forall(x,y) \tag{55}$$

with $\mathsf{U}[\sigma(x,y)] = \mathsf{U}[z(y)] = \frac{1}{2}\int\int \boldsymbol{\sigma}(x,y)\cdot\boldsymbol{\varepsilon}(x,y)\,dx\,dy$ and $\sigma_x(x,y)$, $\sigma_y(x,y)$ and $\tau_{xy}(x,y)$ given by Eqs. (47), (49) and (50).

In the following some results are reported, obtained by applying the shown approach to the considered case.

4.2. Numerical results

For the NRT semi-plane depicted in Fig. 3, one considers the specific weight $\gamma = 2000\ \text{kg/m}^3$ and the load resultant $R = -1000$ kg. For characterizing the function $k(y)$, one assumes a number of coefficients k_i in Eq. (53) equal to $n = 3$ and an initial value $k_0 = k_0(0) = 2$.

The condition $k'(0) = 0$ (i.e. k_1 constrained to be null) is released, in order to allow some shear stress to develop by friction at the interface between the semi-plane and the foundation block that is assumed to transmit the load. Such shear stress is, therefore, identified in solution at the Complementary Energy minimum as a reactive surface force.

The optimal solution is then searched for by looking at the coefficients k_i, for $i = 0, 1, 2, 3$, and β such to solve the problem Eq. (55).

As above mentioned the first coefficient k_0 represents the load trace and is, thus, fixed.

The numerical implementation of the complementary energy problem Eq. (55), gives optimal values $k_1 = 0.1957$, $k_2 = 0.0067$, $k_3 = 0.00008$ and $\beta = 0.0004$.

One can, thus, build up the optimal expression of the function $k(y)$ by Eq. (53) and determine the approximated stress diffusion by means of Eqs. (47), (49) and (50).

The calculated stress distribution is shown in Figs. 4 and 5 along some horizontal and vertical fundamental lines.

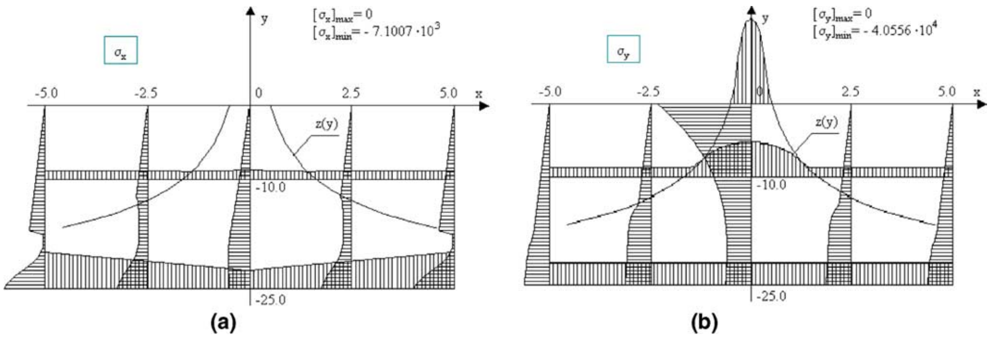

Fig. 4. The stress components along some horizontal and vertical fundamental lines: (a) σ_x; (b) σ_y.

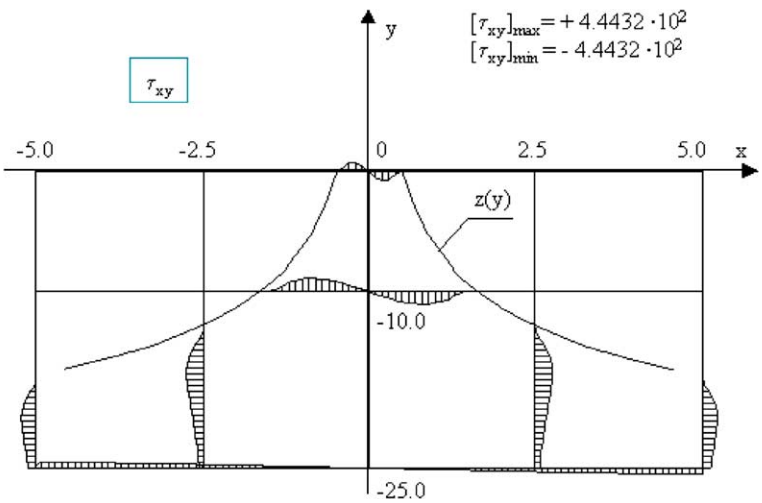

Fig. 5. The stress component τ_{xy} along some horizontal and vertical fundamental lines.

5. Conclusions

In the paper, one approaches theoretical and numerical features of problems relevant to continua that can be modelled by Not Resisting Tension (NRT) models. Actually the NRT material is a phenomenological model able to effectively interpret the behaviour of mechanical bodies made by not-cohesive materials, such as masonries and soils.

After introducing the basic elements of the NRT theory, one gives an analytical demonstration of the main relations governing the NRT behaviour, basically starting from the imposition of Kuhn–Tucker's stationarity conditions for constrained optimisation procedures for NRT continua.

Finally the problem of stress diffusion in an elastic-NRT semi-plane is approached. An equilibrated stress fields subspace is built up, in order to search for an approximate solution of the problem. The numerical results exhibit some features analogous to the experimental stress diffusion in subsoil carrying vertical loads on the free surface.

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